78[L].—Robert Spira, *Table of the Riemann Zeta Function*, ms. volume of 35 typewritten sheets bound with 904 unnumbered computer sheets, with two supplementary volumes of unrounded computer output (3p. + 600 sheets and 3p. + 720 sheets, respectively) deposited in UMT File.

This calculation of the Riemann zeta function $\zeta(\sigma + it)$ is for the values $\sigma = -1.6(.1)2.6$, $t = 0(.1)100.0$, 13D. The limit of 2.6 exceeds the line where $\zeta'(s) \neq 0$. The calculations were carried out with a minimum of 18 significant digits. For uniformity, all numbers are given to 13 decimals. If there are 15 figures or less, one can reasonably expect to have only rare rounding errors. For 16 figures, the accuracy appears to be ± 1 in the last place. There is also a manuscript table giving the unrounded results to 18 significant figures.

The values of the Haselgrove-Miller table were useful as check values, and they appear to be as accurate as claimed. On the $\sigma = \frac{1}{2}$ line, 63 values were found to be off by a unit in the last digit, while on $\sigma = 1$, a total of 79 such values were found. The actual errors were around .5 or .6 units except for Im $\zeta(1 + 90.8i) = -0.075218$ 6942183 versus their -0.075218. The number and size of the errors increased with t. The values of Shafer on the real axis and the value of $\zeta(\frac{1}{2})$ by Gram also served as check values [1, Sec. 22].

A bisection method was used to calculate the first 30 zeros. An independent calculation of the first 50 zeros to 50 places by M. D. Bigg showed small discrepancies in the 16th decimal, the largest being ± 4 in zeros 11, 12, and 23.

The computing was done using the Euler-Maclaurin sum formula method used by Hutchinson [2]. The general method used for checking was the functional equation:

$$\zeta(1 - s) = \left[2 \cos \frac{\pi}{2} s\right] [(2\pi)^{-s}\Gamma(s)]\zeta(s).$$

First $\zeta(s)$ was calculated, and then the reflected value. Then $\zeta(1 - s)$ was calculated and compared. The results of the comparison gave a running account of how the calculation was proceeding. Along the line $\sigma = \frac{1}{2}$, the calculations were performed in two different ways by varying parameters. The coefficients $c_n = -B_{2n+2}/(2n + 2)(2n + 1)B_{2n}$ were calculated using

$$B_{2n} = (-1)^{n+1} (2\pi)^{-2n} 2(2n)!\, \zeta(2n),$$

so

$$c_n = \frac{1}{4\pi^2}\left[\frac{\zeta(2n + 2)}{\zeta(2n)}\right],$$

and the coefficients of the Dirichlet series for the quantity in brackets are easily found.

The complete list of tables is as follows:

I	arcsin x,	x: .1(.1)1.0	18S
IIa	c_n	n: 1(1)7	exact
		n: 8	80D
		n: 9	63D
		n: 10(1)47	30D

IIb	Dirichlet series coefficients $\sum_{d\mid k} d^2\mu(d)$		
		k: 1(1)50	exact
III	$J_n\left(\frac{\pi}{2}\right)$,	n: 0(1)28	30S
		n: 29(1)38	15S
IV	Coefficients for $\cos\frac{\pi}{2}x = \sum a_n x^{2n}$		20D
V	$\sin x$, $\cos x$		
		x: $10^2(10^2)10^3(10^3)10^4(10^4)10^5$	18D
VI	$I_n(.05)$,	n: 0(1)10	31S
VII	Coefficients for $e^{.05x} = 1 + \sum a_n x^n$		20D
VIII	$t = 3 + 2\sqrt{2} - 2\sqrt{4 + 3\sqrt{2}}$		32S
	$4t^n/n$,	n: 1(1)22	30S
IX	$k = 3 + 2\sqrt{2}$		36S
	Coefficients for $\log\left(\frac{1+kz}{1-kz}\right) = \sum a_n z^n$		20D
X	Chebyshev coefficients for $\arcsin\left(\frac{x}{\sqrt{2}}\right) = \sum a_n T_{2n+1}(x)$		
		n: 0(1)30	30S
XI	Coefficients for $\arcsin\left(\frac{x}{\sqrt{2}}\right) = \sum a_n x^{2n+1}$		
		n: 0(1)22	20D
XII	Coefficients for asymptotic expansion of Γ-function		
		n: 1, 2, 3	exact
		n: 4(1)18	31S
	A_n/A_{n+1}	n: 1, 2	exact
		n: 3(1)17	31S
XIII	Chebyshev polynomial coefficients		
	$T_n(x)$,	n: 1(1)97	exact
XIV	$\zeta(\sigma + it)$,	σ: $-1.6(.1)2.6$; t: 0(.1)100	13D

There is also a copy of Bigg's values of the first 50 zeros to 50 decimal places.

A more complete discussion of the calculation and the formulas used is attached to the original tables. There is also a discussion of the theory of the arcsin Chebyshev series, which is rather complicated theoretically. It is planned to have the tables available on microcards.

The planning and programming took about one and a half years. The running time was several hundred hours. Procedures and documentation standards were set up on the basis of advice from Dr. Francis J. Murray of Special Research in Numerical Analysis. That such an involved program could be prepared and running in such a short period of time is certainly a tribute to the powerful and effective tools he supplied. Especially noteworthy was the systematic application of a 2000-command tracer program. The program itself had about 6000 commands, and there were at least another 6000 commands in the auxiliary programs for calculating the constants, testing, etc. The 7072 machine was extremely reliable.

There have been several effects on the theory of the Riemann zeta function as a result of this computation. First of all, zeros of $\zeta'(s)$ were discovered in the critical strip. The evidence of this calculation and further calculations results in the following conjecture: There are zeros of $\zeta'(s)$ on dense set of vertical lines for $\frac{1}{2} < \sigma \leqq 1$, and no zeros of $\zeta'(s)$ for $\sigma \leqq \frac{1}{2}$ except on the negative real axis. A paper is being prepared to show that $\zeta'(s) \neq 0$ in a very large portion of the left half plane $(\sigma \leqq \sigma_0, t \geqq t_0)$.

A second result is that $|\zeta(1-s)| > |\zeta(s)|$ for $\frac{1}{2} < \sigma \leqq 1$, $t \geqq 10$, if $\zeta(s) \neq 0$. Thus, always for this region $|\zeta(1-s)| \geqq |\zeta(s)|$, and the strictness of this inequality is equivalent to the Riemann hypothesis.

AUTHOR'S SUMMARY

1. A. FLETCHER, J. C. P. MILLER, L. ROSENHEAD & L. J. COMRIE, *An Index of Mathematical Tables*, Addison-Wesley, Reading, Massachusetts, 1962.

2. J. I. HUTCHINSON, "On the roots of the Riemann Zeta Function," *Trans. Amer. Math. Soc.*, v. 27, 1925, p. 49–60.

79[L].—(a) MARIYA I. ZHURINA & LENA N. KARMAZINA, *Tablitsy funktsii Lezhandra* $P_{-1/2+i\tau}(x)$, Tom II (*Tables of the Legendre functions* $P_{-1/2+i\tau}(x)$, Vol. II), Akad. Nauk SSSR, Moscow, 1962, iv + 414 p., 27 cm. Price 4.42 rubles.

(b) M. I. ZHURINA & L. N. KARMAZINA, *Tablitsy i formuly dlya sfericheskikh funktsii* $P^m_{-1/2+i\tau}(z)$ (*Tables and formulas for the spherical functions* $P^m_{-1/2+i\tau}(z)$), Akad. Nauk SSSR, Moscow, 1962, xivii + 58 p., 27 cm. Price 0.58 rubles.

Both volumes are in the series of *Mathematical Tables* of the Computational Centre of the Academy of Sciences of the USSR.

(a) This second volume, promised in the first volume and mentioned in the review of that volume (*Math. Comp.*, v. 16, p. 253–254, April 1962, where *for* Karamazina *read* Karmazina and *for* Izdatel'stov *read* Izdatel'stvo), has now been published. It will be remembered that Vol. I was for $x^2 < 1$ and that Vol. II was to be for $x > 1$. Replacing $-\frac{1}{2} + i\tau$ by s for convenience in the whole of the present reviews, the second volume does indeed give values of $P_s(x)$ to 7D without differences for $\tau = 0(0.01)50$ and $x = 1.1(0.1)2(0.2)5(0.5)10(10)60$. The main table occupies pages 11–270, i–iv, 271–407, a total of 401 pages. In principle there are four pages for each of the hundred ranges of width 0.50 in τ, but the table for $\tau = 32.50(0.01)33.00$ occupies five pages, the material having been skillfully and hardly noticeably spaced out (presumably to retrieve an error in pagination).

On pages 408–413 is an auxiliary table which for $x = 1.01(0.01)3(0.05)5(0.1)10$ gives, to 7D without differences, values of $\theta = \cosh^{-1}x$ and of the first four coefficients in the expansion of $P_s(\cosh\theta)$ in multiples of $\tau^{-n}J_n(\tau\theta)$. The values of θ have been read against the Harvard 9D tables [1], and appear to be correct on the convention that rounding is always downward, except that upward rounding occurs at $x = 1.61, 1.68, 1.72, 1.83, 2.00, 4.45, 7.60$. Nine decimals are not enough to decide at $x = 2.04$, but special calculation shows that upward rounding occurs here also.

(b) In this slim volume, which relates to both $x^2 < 1$ and $x > 1$, the same authors give first, on pages v–xxxviii, a collection of formulas relating to $P_s{}^m(z)$. Then follow a description of the tables and a bibliography of 43 items.